Rapid Calculation of the Resonance Frequency for Rotationally Restrained Rectangular Plates

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An approximate solution for the vibration of rectangular plates with edges subjected to elastic torsional restraints is being developed. The solution is in closed form and accounts for the general case of unequal restraints on all edges. It is explicit in terms of the restraint parameters and geometric ratio of the plate. The Rayleigh-Ritz method is used with a trial function consisting of the product of specially developed expressions approximating the deflection shape of vibration of a rotationally restrained beam. The results are compared to solutions obtained by more elaborate analyses. The discrepancy is no more than 0.5% for a wide range of edge restraint parameters and geometric ratios.

Nomenclature

a,b = plate length and width
D = plate bending stiffness, $D=Eh^3/12(1-\nu^2)$
h = plate thickness
k = torsional spring constant (dimension of moment/length)
m = mass per unit area of the plate
K = torsional spring constant (dimension of moment)
R = plate geometrical ratio, $0 \leq (R=b/a) \leq 1$
T = kinetic energy
U = strain energy
W = plate deflection function
x,y = rectangular coordinate system
β = dimensionless torsional restraint parameter, $\beta=kL/EI$ for a bar, $\beta=ka/D$ on plate edges parallel to the y axis, and $\beta=kb/D$ on edges parallel to the x axis, L is the bar length
μ,λ = frequency parameters for a bar defined as μ^2 (or λ^2) $=\omega\sqrt{mL4/EI}$
μ_1,μ_2,μ_3 = frequency parameters for pinned-pinned, pinned-clamped, and clamped-clamped bars, respectively
χ = x/a
ψ = y/b
ω = circular resonance frequency

Introduction

EXCEPT for the case of simply supported edges, exact solutions for the vibrational characteristics of rectangular plates are not known. Approximate solutions for plates with edges subjected to various combinations of pinned, clamped, and free types of support are available in the literature.[1-4] A comprehensive survey of such solutions is presented in Ref. 5. On the other hand, there are only few attempts to obtain a solution for the vibration of the rectangular plate when the edges are elastically restrained against rotation.

One of the earliest attempts to deal with such a problem was made for the purpose of investigating the mechanical vibration of a chassis used for electronic equipment.[6] Typically, such a chassis is formed by bending the edges of a flat plate. In Ref. 6, the chassis was treated as a plate with elastic edge supports. The Rayleigh-Ritz[7] method was used and a trial function was assumed to be an arithmetic average of the eigenfunctions for plates with simply supported edges and those having clamped edges. Analytical and experimental results were obtained for a particular chassis.

Other investigators dealt with the vibration of rotationally restrained rectangular plates. Carmichael[9] considered the case of equal restraints on opposite edges. The Ritz method was used and the trial function was assumed as the sum of a series whose terms consisted of the products of the mode shapes of bars elastically restrained in a manner similar to the plate edges. The process of minimizing the energy equation was extremely laborious and numerical results were possible to obtain only at specific values of the restraint parameters and plate geometrical ratio. Simplifying assumptions were introduced and an approximate frequency equation was derived for use in connection with numerical tables. The Ritz method was also used in Ref. 8 to obtain results for the case of an equally restrained square plate. The lowest three frequency parameters were calculated and the results were presented graphically as a function of the restraint parameter. Other results are presented in Ref. 5. Numerical values are given for the rectangular plate with two opposite edges simply supported and the other edges elastically restrained. These are tabulated for various restraint degrees and plate aspect ratios.

None of the known solutions dealt with the general case of unequal restraints on all edges. Moreover, these solutions require elaborate calculations to the extent that their practical use is limited to the published numerical values.

The purpose of this paper is to develop a closed-form type solution for the general case of the problem.

Method of Solution

Approximate solutions for the vibrational frequencies of structural elements may be derived using Rayleigh's principle.[10,11] In such a method, it is necessary to start by assuming a displacement function which satisfies the geometric boundary conditions of the problem. The approximate frequency is determined by equating the strain energy of the structure U to its kinetic energy T; that is,

$$U=T \tag{1}$$

For the case of a thin isotropic homogeneous plate elastically restrained against rotation, the strain and kinetic energies are given by

$$U=\frac{D}{2}\int_0^b\int_0^a\left\{\left(\frac{\partial^2 W}{\partial x^2}\right)^2+\left(\frac{\partial^2 W}{\partial y^2}\right)^2+2\frac{\partial^2 W}{\partial y^2}\frac{\partial^2 W}{\partial y^2}\right.$$
$$\left.-2(1-\nu)\left[\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}-\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right]\right\}\mathrm{d}x\mathrm{d}y$$

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Table 1 Evaluation of integrals

$f(x)$	$\int_0^L f(x)\,\mathrm{d}x$
$\cos^2 Qx/L$	$\left(\frac{1}{2}+\frac{1}{4Q}\sin 2Q\right)L$
$\sin^2 Qx/L$	$\left(\frac{1}{2}-\frac{1}{4Q}\sin 2Q\right)L$
$\cosh^2 Qx/L$	$\left(\frac{\sinh 2Q}{4}+\frac{Q}{2}\right)\frac{L}{Q}$
$\sinh^2 Qx/L$	$\left(\frac{\sinh 2Q}{4}-\frac{Q}{2}\right)\frac{L}{Q}$
$\cos(Q_1x/L)\cos(Q_2x/L),\quad Q_1\neq Q_2$	$(Q_2\cos Q_1\sin Q_2-Q_1\sin Q_1\cos Q_2)L/(Q_2^2-Q_1^2)$
$\sin(Q_1x/L)\sin(Q_2x/L),\quad Q_1\neq Q_2$	$(Q_1\cos Q_1\sin Q_2-Q_2\sin Q_1\cos Q_2)L/(Q_2^2-Q_1^2)$
$\cosh(Q_1x/L)\cosh(Q_2x/L),\quad Q_1\neq Q_2$	$(Q_2\cosh Q_1\sinh Q_2-Q_1\sinh Q_1\cosh Q_2)L/(Q_2^2-Q_1^2)$
$\sinh(Q_1x/L)\sinh(Q_2x/L),\quad Q_1\neq Q_2$	$(Q_2\sinh Q_1\cosh Q_2-Q_1\cosh Q_1\sinh Q_2)L/(Q_2^2-Q_1^2)$
$\cos(Q_1x/L)\sin(Q_2x/L),\quad Q_1\neq Q_2$	$(Q_2-Q_2\cos Q_1\cos Q_2-Q_1\sin Q_1\sin Q_2)L/(Q_2^2-Q_1^2)$
$\cos(Q_1x/L)\cosh(Q_2x/L)$	$(Q_1\cosh Q_2\sin Q_1+Q_2\cos Q_1\sinh Q_2)L/(Q_1^2+Q_2^2)$
$\cos(Q_1x/L)\sinh(Q_2x/L)$	$(Q_1\sin Q_1\sinh Q_2-Q_2+Q_2\cos Q_1\cosh Q_2)L/(Q_1^2+Q_2^2)$
$\sin(Q_1x/L)\cosh(Q_2x/L)$	$(Q_1-Q_1\cos Q_1\cosh Q_2+Q_2\sin Q_1\sinh Q_2)L/(Q_1^2+Q_2^2)$
$\sin(Q_1x/L)\sinh(Q_2x/L)$	$(Q_2\sin Q_1\cosh Q_2-Q_1\cos Q_1\sinh Q_2)L/(Q_1^2+Q_2^2)$
$\cosh(Q_1x/L)\sinh(Q_2x/L),\quad Q\neq Q_2$	$(Q_2\cosh Q_1\cosh Q_2-Q_2-Q_1\sinh Q_1\sinh Q_2)L/(Q_2^2-Q_1^2)$
$\cos(Qx/L)\sin(Qx/L)$	$(L/2Q)\sin^2 Q$
$\cosh(Qx/L)\sinh(Qx/L)$	$(L/2Q)(\cosh^2 Q-1)$

$$+\frac{1}{2}\left\{\int_0^b\left[k_1\left(\frac{\partial W}{\partial x}\right)^2_{x=0}+k_2\left(\frac{\partial W}{\partial x}\right)^2_{x=a}\right]\mathrm{d}y\right.$$
$$\left.+\int_0^a\left[k_3\left(\frac{\partial W}{\partial y}\right)^2_{y=0}+k_4\left(\frac{\partial W}{\partial y}\right)^2_{y=b}\right]\mathrm{d}x\right\} \tag{2}$$

and

$$T=\frac{m\omega^2}{2}\int_0^b\int_0^a W^2\mathrm{d}x\mathrm{d}y \tag{3}$$

The Ritz method[7] permits simultaneous approximate determination of the first n resonance frequencies of a vibrating system. This method was first used in the study of vibration of a square plate with free edges. The deflection function was assumed in the form

$$W(x,y)=\sum_{n=1}^{N}\sum_{m=1}^{M}a_{mn}X_m(x)\,Y_n(y) \tag{4}$$

where $X_m(x)$ and $Y_n(x,y)$ are the mth and nth modes of vibration of the free-free bars.

Later, the Ritz method was also used to develop solutions for plates with various other boundary conditions. The degree of accuracy attained by the method depends on the choice of the approximating function $W(x,y)$ and the number of terms in the assumed series. There is ample evidence however, that with careful choice of the approximating function a one-term representation of the mode shape may be used without great loss of the accuracy of the resulting solution. This is clearly observed in the results obtained by Young[12] for a square plate. A 36-term series of the type given in Eq. (4) was considered, the first six resonance frequencies were obtained, and the relative magnitudes of the coefficients of each term in the series were calculated. The results show that for an mnth mode the predominant coefficient in the series is that of the mnth term; the relative magnitudes of the coefficients of the other terms are no more than 1% that of the mnth term. Some of the coefficients could be as high as 15% depending on how far the nodal lines deviate from being straight or from being parallel to the edges of the plate. For those modes described as $mn\pm nm$, the nodal lines are no longer parallel to the edges of the plate and in such cases the relative magnitudes of the coefficients of the mnth and nmth terms are of the same order of magnitude. Even in such instances, a choice of the deflection function consisting of an mnth term only was shown to yield estimates of the natural frequency of high accuracy.[1] This issue was also studied in detail by Leissa.[14] The average difference in the frequency parameters when calculated by 1- and 36-term solutions was found not to exceed 0.5% for a rectangular plate with combinations of simply supported and clamped edges.

Analysis

The choice of the deflection function not only influences the accuracy of the final result but also determines the amount of labor expended in deriving the solution and, more importantly, the form of the final solution. For the elastically restrained plates, a choice of the deflection as an explicit function of the edge restraint parameters will result in a closed-form solution for the frequency of vibration.

In the present analysis, the deflection function will be chosen to consist of a single term in the form:

$$W_{mn}(x,y)=X_m(x)\,Y_n(y) \tag{5}$$

where $X_m(x)$ and $Y_n(y)$ are the mode shapes of vibration of bars elastically restrained against rotation in a manner similar to the edges of the plate.

Considering the fundamental frequency of vibration, m and n are set equal to unity and the mode shapes of the bars are taken in the explicit approximate form developed in Ref. 13. For the bar parallel to the x axis the mode shape is:

$$X(x) = \frac{6.795 W_{00}(x) + \beta_1 W_{\infty 0}(x) + \beta_2 W_{0\infty}(x) + 0.1403\beta_1\beta_2 W_{\infty\infty}(x)}{6.795 + \beta_1 + \beta_2 + 0.1403\beta_1\beta_2} \tag{6}$$

where

$$W_{00}(x) = C_1 \sin\mu_1\chi, \quad W_{\infty 0}(x) = C_2[\cosh\mu_2\chi - \cos\mu_2\chi + q_1(\sin\mu_2\chi - \sinh\mu_2\chi)]$$

$$W_{0\infty}(x) = C_3[\sin\mu_2\chi + q_2\sinh\mu_2\chi], \quad W_{\infty\infty}(x) = C_4[\cosh\mu_3\chi - \cos\mu_3\chi + q_3(\sin\mu_3\chi - \sinh\mu_3\chi)]$$

are the exact mode shapes of a pinned-pinned, clamped-pinned, pinned-clamped, and built-in beams, respectively, and

$$0 \leq (\chi = x/a) \leq 1; \quad \mu_1 = \pi, \quad \mu_2 = 3.927, \quad \mu_3 = 4.73$$

$$q_1 = 1.0008, \quad q_2 = 0.02786, \quad q_3 = 0.9825; \quad c_1 = 1, \quad c_2 = 0.6921, \quad C_3 = 0.9793, \text{ and } C_4 = 0.6296$$

For the bar parallel to the y axis, the mode shape is the same format as Eq. (6) where χ, β_1, and β_2 are replaced by ψ, β_3 and β_4, respectively.

$$Y(y) = [6.795 W_{00}(y) + \beta_4 W_{\infty 0}(y) + \beta_3 W_{0\infty}(y) + 0.1403\beta_3\beta_4 W_{\infty\infty}(y)]/[6.795 + \beta_3 + \beta_4 + 0.1403\beta_3\beta_4] \tag{7}$$

The explicit forms of $X(x)$ and $Y(y)$ were obtained[13] by assuming the deflection shape of the elastically restrained bar to be the weighted sum of the four terms W_{00}, $W_{\infty 0}$, $W_{0\infty}$ and $W_{\infty\infty}$ with coefficient functions of the restraint parameters only. The coefficients were determined by forcing the assumed fucntion to satisfy the boundary conditions of the elastically restrained bar.

Equations (6) and (7) are used to evaluate the energy terms given by Eqs. (2) and (3). The integrals appearing in those terms are listed in Table 1 and numerically evaluated in Table 2. Thus the fundamental frequency of vibration is determined using Eq. (1) as follows:

$$\omega = \sqrt{\{D[(U_0U_1 + U_6 + U_7)R^4 + 2U_4U_5R^2 + U_2U_3 + U_8 + U_9]\}/mb^4U_1U_3} \tag{8}$$

where

$$U_0 = 4694 + 8\beta_1^2\beta_2^2 + 238(\beta_1^2 + \beta_2^2) + 251\beta_1\beta_2 + 1319(\beta_1 + \beta_2) + 58.4(\beta_1^2\beta_2 + \beta_1\beta_2^2)$$

$$U_1 = 48.2 + 0.016\beta_3^2\beta_4^2 + 1.007(\beta_3^2 + \beta_4^2) + 3.56\beta_3\beta_4 + 13.6(\beta_3 + \beta_4) + 0.245(\beta_3^2\beta_4 + \beta_3\beta_4^2)$$

$U_2 \equiv U_0$ by replacing β_1 and β_2 by β_3 and β_4, respectively

$U_3 \equiv U_1$ by replacing β_3 and β_4 by β_1 and β_2, respectively

$$U_4 = -475.6 - 0.193\beta_1^2\beta_2^2 - 11.55(\beta_1^2 + \beta_2^2) - 32.8\beta_1\beta_2 - 134(\beta_1 + \beta_2) - 2.62(\beta_1^2\beta_2 + \beta_1\beta_2^2)$$

$U_5 \equiv U_4$ by replacing β_1 and β_2 by β_3 and β_4, respectively

$$U_6 = \beta_1 U_1[30.84 + 5.68\,\beta_2]^2, \quad U_7 = \beta_2 U_1[30.84 + 5.68\,\beta_1]^2, \quad U_8 = \beta_3 U_3[30.84 + 5.68\,\beta_4]^2, \quad U_9 = \beta_4 U_3[30.84 + 5.68\,\beta_3]^2$$

Table 2 Numerical values of the integrals

$f(x)$	$\frac{1}{a}\int_0^a f(x)\,^{a}dx$ when Q_1 and Q_2 take the values:						
	μ_1 and μ_1	μ_2 and μ_2	μ_3 and μ_3	μ_2 and μ_1	μ_3 and μ_1	μ_2 and μ_3	μ_3 and μ_2
$\cosh(Q_1x/a)\cosh(Q_2x/a)$	—	82.4971	339.7146	—	—	166.613	166.613
$\cosh(Q_1x/a)\sinh(Q_2x/a)$	—	81.9333	339.16	—	—	166.212	165.789
$\cosh(Q_1x/a)\cos(Q_2x/a)$	—	−4.5695	−5.88223	—	—	−3.13032	−9.175
$\cosh(Q_1x/a)\sin(Q_2x/a)$	—	0.12905	−5.98649	3.27775	5.61742	−2.56618	−0.74652
$\sinh(Q_1x/a)\sinh(Q_2x/a)$	—	81.4971	338.715	—	—	165.502	165.502
$\sinh(Q_1x/a)\cos(Q_2x/a)$	—	−4.69682	−5.98699	—	—	−3.23172	−9.30029
$\sinh(Q_1x/a)\sin(Q_2x/a)$	—	−0.00182	−6.09311	3.1511	5.519	−2.69334	−0.85186
$\cos(Q_1x/a)\cos(Q_2x/a)$	—	0.56366	0.49814	—	—	0.48809	0.48809
$\cos(Q_1x/a)\sin(Q_2x/a)$	—	0.063663	0.10568	−0.16576	−0.25569	0.289502	−0.09088
$\sin(Q_1x/a)\sin(Q_2x/a)$	0.50	0.43634	0.50186	0.40014	0.25122	0.40785	0.40786

[a] The same values apply to $\frac{1}{b}\int_0^b f(y)\,dy$.

Table 3 Comparison with results of Ref. 9 for $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 20$

R	1	0.8	0.6	0.4	0.2
Ref. 9	31.09	25.80	22.30	20.30	19.38
Eq. (8)	31.084	25.80	22.30	20.305	19.371
Discrepancy, %	−0.02	0.00	0.00	0.02	−0.04

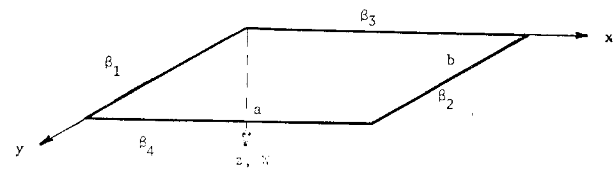

Fig. 1 Elastically restrained rectangular plate.

Table 4 Comparison with the approximate equation of Ref. 9

		R											
		0.4			0.6			0.8			1.0		
$\beta_1 = \beta_2$	$\beta_3 = \beta_4$	a	b	c	a	b	c	a	b	c	a	b	c
0	0	11.4488	11.4470	−0.02	13.4227	13.4204	−0.02	16.1862	16.1833	−0.02	19.7393	19.7357	−0.02
0	1	12.9293	12.9175	−0.09	14.7083	14.6963	−0.08	17.2702	17.2579	−0.07	20.6406	20.6280	−0.06
0	5	16.285	16.2562	−0.18	17.7569	17.7276	−0.16	19.9637	19.9340	−0.15	22.9786	22.9486	−0.13
0	1000	23.1908	23.1626	−0.12	24.4197	24.3738	−0.19	26.293	26.2254	−0.26	28.9122	28.8214	−0.32
1	0	11.4915	11.4886	−0.02	13.6003	13.5951	−0.04	16.6428	16.6344	−0.05	20.6406	20.6280	−0.06
1	1	12.9672	12.9543	−0.1	14.8706	14.856	−0.10	17.6990	17.6817	−0.1	21.5043	21.4834	−0.1
1	5	16.3152	16.2855	−0.18	17.8917	17.8604	−0.18	20.3358	20.3021	−0.16	23.7576	23.7206	−0.16
1	1000	23.2123	23.183	−0.13	24.5185	24.4704	−0.2	26.5777	26.5065	−0.27	29.5364	29.4401	−0.33
5	0	11.6317	11.6263	−0.05	14.1162	14.1051	−0.08	17.884	17.8646	−0.11	22.9786	22.9486	−0.13
5	1	13.0917	13.0766	−0.12	15.344	15.3241	−0.13	18.8709	18.8437	−0.14	23.7576	23.7206	−0.16
5	5	16.4148	16.3835	−0.19	18.2882	18.2525	−0.2	21.3654	21.3232	−0.2	25.8172	25.766	−0.2
5	1000	23.2859	23.2552	−0.13	24.8166	24.7647	−0.21	27.3853	27.3069	−0.29	31.2333	31.1260	−0.34
1000	0	12.1993	12.1672	−0.26	15.775	15.7185	−0.36	21.3455	21.2693	−0.36	28.9122	28.8214	−0.32
1000	1	13.599	13.5601	−0.29	16.8834	16.8216	−0.37	22.1801	22.0988	−0.37	29.5364	29.4401	−0.33
1000	5	16.8270	16.776	−0.30	19.6069	19.5346	−0.37	24.351	24.2591	−0.38	31.2333	31.1260	−0.34
1000	1000	23.6088	23.5580	−0.22	25.8664	25.7753	−0.35	29.8697	29.7351	−0.45	35.9652	35.7922	−0.5

[a] Values of Ref. 9. [b] Values of Eq. (8). [c] Percent difference between b and a.

Table 5 Comparison with the results of Ref. 5 for $\beta_1 = \beta_2 = \beta$ and $\beta_3 = \beta_4 = 0$

		R				
β		0.4	0.6	0.8	0.9	1.0
	a	11.4487	13.4229	16.1871	17.8649	19.7372
0	b	11.447	13.4204	16.1833	17.8608	19.7357
	c	−0.02	−0.02	−0.02	−0.02	−0.01
	a	11.4882	13.5956	16.6391	18.5198	20.639
1	b	11.4882	13.5956	16.6344	18.5114	20.628
	c	0.00	0.00	−0.03	−0.04	−0.05
	a	11.5268	13.7443	17.0188	19.0615	21.3763
2	b	11.5275	13.7478	17.0148	19.0527	21.3616
	c	0.00	0.02	−0.02	−0.05	−0.07
	a	11.6103	14.0958	17.8659	20.2516	22.9667
5	b	11.6263	14.1051	17.8646	20.2411	22.9486
	c	0.14	0.07	−0.01	−0.05	−0.08
	a	11.7119	14.4738	18.7263	21.4291	24.5136
10	b	11.7429	14.4892	18.7286	21.4242	24.501
	c	0.26	0.11	0.01	−0.02	−0.05
	a	11.8363	14.8741	19.5950	22.5976	26.0228
20	b	11.8751	14.8948	19.5988	22.5957	26.0168
	c	0.33	0.14	0.02	0.00	−0.08
	a	11.9039	15.0825	20.0292	23.175	26.7646
30	b	11.9464	15.1036	20.0333	23.174	26.7582
	c	0.36	0.14	0.02	0.00	−0.02
	a	11.9770	15.2949	20.4664	23.7501	27.4953
50	b	12.0205	15.3154	20.4658	23.7458	27.4874
	c	0.36	0.13	0.00	−0.02	−0.03
	a	12.0471	15.4931	20.8616	24.2693	28.153
100	b	12.0905	15.5098	20.8563	24.2590	28.1387
	c	0.36	0.11	−0.03	−0.04	−0.05
	a	—	15.6821	21.2363	24.7553	28.764
500	b	12.158	15.6937	21.2204	24.7350	28.7403
	c	—	0.07	−0.07	−0.08	−0.08

[a] Corrected values of Ref. 5. [b] Values of Eq. (8). [c] Percent difference between b and a.

Table 6 Comparison with the results of Ref. 14 for combinations of simply supported and clamped conditions

					R		
β_1	β_2	β_3	β_4		0.4	2/3	1
				a	11.4487	14.2561	19.7392
0	0	0	0	b	11.447	14.2537	19.7357
				c	−0.02	−0.02	−0.02
				a	23.2774	25.0436	28.9509
0	0	∞	∞	b	23.2503	25.0006	28.9038
				c	−0.12	−0.17	−0.16
				a	12.1347	17.3730	
∞	∞	0	0	b	12.1767	17.3767	
				c	0.35	0.02	”
				a	11.7502	15.5783	23.6463
∞	0	0	0	b	11.8174	15.6419	23.666
				c	0.57	0.41	0.08
				a	16.6276	18.9012	
0	0	∞	0	b	16.5847	18.8728	
				c	−0.26	−0.15	”
				a	23.648	27.010	35.992
∞	∞	∞	∞	b	23.6507	26.9833	35.9304
				c	0.01	−0.1	−0.17
				a	23.440	25.861	31.829
∞	0	∞	∞	b	23.4589	25.8778	31.8284
				c	0.08	0.06	0.00
				a	17.1312	21.4076	
∞	∞	∞	0	b	17.1294	21.4030	
				c	−0.01	−0.02	”
				a	16.849	19.952	27.056
∞	0	∞	0	b	16.8669	19.9992	27.1256
				c	0.11	0.24	0.26

[a] Values of Ref. 14. [b] Values of Eq. (8). [c] Percent difference between (b) and (a). [d] In Eq. (8), $\beta = 10^6$ for clamped edge conditions. [e] Results of Ref. 14 are exact for the first five cases, the rest of the cases are results of 36-term solution.

Table 7 Comparison of the accuracy of the solution at selected values of the restraint parameters and geometric ratio

R	β_1	β_2	β_3	β_4	Solution of Appendix A	Eq. (9)	Discrepancy, %
	0	1	1	5	22.1815	22.1598	−0.1
	0	0	5	1000	25.6479	25.5916	−0.22
	0	0	1000	1000	28.9118	28.8214	−0.31
1	1	5	1	5	23.6385	23.6052	−0.14
	5	1000	5	1000	30.4926	30.3941	−0.32
	5	5	1000	1000	31.2327	31.1260	−0.34
	1000	0	1000	0	27.1584	27.0729	−0.31
	0	1	1	5	18.7743	18.7530	−0.11
	0	0	5	1000	22.8115	22.7603	−0.22
	0	0	1000	1000	26.2926	26.2254	−0.26
0.8	1	5	1	5	19.5108	19.4830	−0.14
	5	1000	5	1000	25.2769	25.1973	−0.31
	5	5	1000	1000	27.3848	27.3069	−0.28
	1000	0	1000	0	22.4218	22.3516	−0.31
	0	1	1	5	16.2639	16.2424	−0.13
	0	0	5	1000	20.7584	20.7124	−0.22
	0	0	1000	1000	24.4192	24.3738	−0.18
0.6	1	5	1	5	16.5534	16.5290	−0.15
	5	1000	5	1000	21.7558	21.6932	−0.29
	5	5	1000	1000	24.8161	24.7647	−0.21
	1000	0	1000	0	19.0398	18.9811	−0.31
	0	1	1	5	14.5844	14.5626	−0.15
	0	0	5	1000	19.4003	19.3587	−0.21
	0	0	1000	1000	23.1903	23.1626	−0.12
0.4	1	5	1	5	14.6598	14.6369	−0.16
	5	1000	5	1000	19.6852	19.6362	−0.25
	5	5	1000	1000	23.2854	23.2552	−0.13
	1000	0	1000	0	16.8844	16.8338	−0.3
	0	1	1	5	13.6392	13.6170	−0.16
	0	0	5	1000	18.6383	18.5999	−0.21
	0	0	1000	1000	22.504	22.4878	−0.07
0.2	1	5	1	5	13.6483	13.6259	−0.16
	5	1000	5	1000	18.6819	18.6416	−0.22
	5	5	1000	1000	22.5143	22.4976	−0.07
	1000	0	1000	0	15.739	15.6932	−0.29

Comparison

Equation (8) is an explicit function of the restraint parameters and geometric ratio. The expression is compared to other solutions in the literature. Comparison with the published results of Carmichael's[9] 36-term solution for equal restraints on all edges is shown in Table 3. Comparison with his approximate formula for equal restraint on opposite edges is given in Table 4. The maximum discrepancy is of the order of 0.5%. Table 5 shows a comparison with the results of Ref. 5 (after E.E. Lundquist) for the case of two opposite edges simply supported and the other two elastically restrained. In all cases, the difference is well below 0.5%.

The expression is also compared to the numerical values of Leissa[14] which are obtained by exact methods and by 36-term series solutions for combinations of simply supported and clamped-edge conditions. The results are given in Table 6 and the maximum error is found to be 0.57% for the case of $R=0.4$ and $\beta_1=\infty$ and $\beta_2=\beta_3=\beta_4=0$.

Because of the scarcity of numerical data for cases of unequal restraints on all edges, it was necessary to generate an independent solution for the purpose of comparison. In the Appendix, a solution is developed using the Ritz method and assuming the deflection function to be a single term consisting of the product of the exact mode shapes of elastically restrained bars. A comparison to this solution is shown in Table 7 for a range of edge restraints and geometric ratios. It is seen that the results of Eq. (8) are within 0.5% of the solution over the range of variables.

Although Eq. (8) is derived using Rayleigh's method, it is interesting to observe that the results, in some instances are lower than the exact values (e.g., Table 6). This is due to the fact that the numbers in the terms of the equation have been rounded off for simpler presentation.

Conclusion

A solution is derived for the fundamental resonance frequency of a rotationally restrained rectangular plate using the Rayleigh-Ritz method and a specially developed trial function. The solution is explicit in the edge restraint parameters and the geometric ratio of the plate. Its accuracy is comparable to that of other solutions obtained by more elaborate analyses. Its closed-form type should be of interest to scientists and engineers.

Appendix—A Ritz Solution Using a Single-Term Exact Mode Shape Assumption

The deflection function is assumed in the form of Eq. (5). The terms $X_m(x)$ and $Y_n(y)$ are taken to be the exact resonance modes of rotationally restrained bars. For the fundamental frequency $m=n=1$ the modes are given by[13]:

$$X(x)=A\cosh\mu\chi+B\sinh\mu\chi+C\cos\mu\chi+D\sin\mu\chi \qquad \text{(A1)}$$

$$Y(y)=E\cosh\lambda\psi+F\sinh\lambda\psi+G\cos\lambda\psi+H\sin\lambda\psi \qquad \text{(A2)}$$

where

$$\chi=x/a, \quad -\tfrac{1}{2}\leq\chi\leq\tfrac{1}{2}$$

$$\psi=y/b, \quad -\tfrac{1}{2}\leq\psi\leq\tfrac{1}{2}$$

$A = \frac{1}{2}\sin\mu$

$B = (-\frac{1}{2}\sin\mu)\gamma$

$C = -\sin\mu/2\cosh\mu/2$

$D = (\cos\mu/2\sinh\mu/2)\gamma$

$$\gamma = \frac{(\beta_2 - \beta_1)\left(\sinh\mu/2 + \dfrac{\cosh\mu/2\ \sin\mu/2}{\cos\mu/2}\right)}{4\mu\sinh\mu/2 + (\beta_1 + \beta_2)\left(\cosh\mu/2 - \dfrac{\cos\mu/2\ \sinh\mu/2}{\sin\mu/2}\right)}$$

E, F, G, and H are equivalent to A, B, C, and D by replacing μ, β_1, and β_2 by λ, β_3, and β_4, respectively.

μ is the lowest frequency parameter of the bar with rotational restraints β_1 and β_2 obtained by solving the following characteristic equation numerically:

$$2\mu^2\sin\mu\ \sinh\mu + \mu(\beta_1 + \beta_2)(\sin\mu\ \cosh\mu - \cos\mu\ \sinh\mu)$$
$$+\beta_1\beta_2(1 - \cos\mu\ \cosh\mu) = 0$$

λ the lowest frequency parameter of the bar with restraints β_3 and β_4. It is obtained from the preceding equation by replacing μ, β_1, and β_2 by λ, β_3, and β_4, respectively. Equations (A1) and (A2) are used to evaluate the strain and kinetic energy terms given by Eqs. (2) and (3) and the plate frequency is obtained from Eq. (1). The solution is programmed for numerical computation.

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